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On the embedding genus distribution of ladders and crosses

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ABSTRACT

In this work, the relations between ladder surface sets and cross surface sets are found. The embedding genus distribution of ladders can be obtained by using the genus distribution of cross type surface sets.

1. Introduction

Throughout this work a surface, a graph and an embedding always imply an orientable cycle, a connected graph and an orientable embedding. The concepts can be found in [2,3,8].

A *linear sequence* is a letter sequence with the relation $\prec$. Since an orientable closed surface can be regarded as forming by gluing the edges of a directed polygon as a direction, a surface can be regarded as an *orientable cycle* S which contains one a and one a^- for each $a \in S$. $\gamma(S)$ denotes the genus of the surface S and $\mathcal{S}$ denotes the set containing all of the surfaces. An equivalence $\sim$ (for example [2]), defined on $\mathcal{S}$, is as follows:

Op1. $AB \sim (Ax)(x^-B)$ where $AB \in \mathcal{S}$ and $x \notin AB$;

Op2. $Ax_1x_2Bx_2^-x_1^- \sim AxBx^- = Ax^-Bx$ where $Ax_1x_2Bx_2^-x_1^- \in \mathcal{S}$ and $x \notin AB$;

Op3. $Axx^-B \sim AB$ where $Axx^-B \in \mathcal{S}$ and $AB \neq \emptyset$.

Lemma 1 (*For Example [9]*). *Let A, B, C and D be linear sequences and let $xABx^-CD$ be a surface. Then*

$$xABx^-CD \sim xBAx^-CD \sim xABx^-DC$$

where $x, x^- \notin ABCD$.

Let U be a surface set. The *genus distribution* of U is

$$g_0(U), g_1(U), g_2(U), \ldots$$

The *genus polynomial* of U is $f_U(x) = \sum_{i=0}^{\infty} g_i(U)x^i$ where $g_i(U)$ denotes the number of distinct surfaces of U with genus i ($i \geq 0$). Given a graph G and a surface S, if there is a homeomorphism $\phi : G \to S$ such that each connected component of $S - \phi(G)$ is homeomorphic to an open disc, then G has a two-cell embedding on S. The *genus of a graph G* is the minimum

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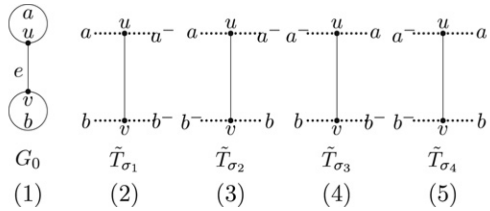

Fig. 1. G_0 and its four joint trees.

genus of the surface which it can be embedded on. The *embedding genus distribution* of G, also called the *genus distribution*, is

$$g_0(G), g_1(G), g_2(G), \ldots.$$

The *embedding polynomial*, also called the *genus polynomial*, of a graph G is $f_G(x) = \sum_{i=0}^{\infty} g_i(G)x^i$ where $g_i(G)$ denotes the number of distinct embeddings of G with genus i. Since determining the genus of a graph is NP-complete [6], it is NP-complete to determine the embedding genus distribution of a graph.

Given a graph G, a *rotation* at a vertex v of G is a cyclic permutation of edges incident with v. A *rotation system* of G is obtained by assigning a rotation at each vertex of G. Let T be a spanning tree of G. A *joint tree* $\tilde{T}$ is formed by splitting each cotree edge a into two semi-edges a and a^-. Given a spanning tree, a joint tree is determined by a rotation system and the associated embedding surface is a cyclic permutation which is formed by the semi-edges and which is determined by the joint tree. For example, four joint trees $\tilde{T}_{\sigma_k}(1 \le k \le 4)$ of G_0 are obtained by letting a and b of G_0 be cotree edges and letting each vertex have a clockwise rotation (see Fig. 1). Embedding surfaces of $\tilde{T}_{\sigma_k}$ for $i = 1, 2, 3$ and 4 are respectively aa^-b^-b, aa^-bb^-, a^-ab^-b and a^-abb^-.

We obtained explicit expressions for the genus distribution for ladder surface sets and cross surface sets [7,9,10]. In this work we get the relations between genera of ladder surface sets and genera of cross surface sets. Since the embedding genus distribution of ladders and crosses can be calculated by using the genus distribution for ladder surface sets and cross surface sets respectively [9,10], the embedding genus distribution of ladders can be obtained by using the genus distribution of cross surface sets. Consequently, explicit expressions for genus distribution for closed-end ladders [1], Ringel ladders [5], circular ladders and Möbius ladders [4] are deduced.

2. Main theorem

Let e_1 and e_2 be edges of a graph G. A *ladder* GL_n is obtained by adding n $(n \ge 1)$ vertices $u_1, u_2, u_3, \ldots, u_n$ on e_1 in sequence, n vertices $v_1, v_2, v_3, \ldots, v_n$ on e_2 in sequence and edges u_lv_l such that u_1v_1 and u_2v_2 are parallel edges. A *cross* GC_n is obtained by adding n vertices $u_1, u_2, u_3, \ldots, u_n$ on e_1 in sequence, n vertices $v_1, v_2, v_3, \ldots, v_n$ on e_2 in sequence and edges u_lv_l such that u_1v_1 and u_2v_2 are not parallel edges. Denote u_lv_l by a_l for $1 \le l \le n$.

Suppose that a_l are distinct letters for $l \ge 1$. The *ladder surface sets* S_k^n are as follows for $1 \le k \le 11$:

$$S_1^n = \{R_1^nR_2^nR_3^nR_4^n\} \qquad S_2^n = \{R_1^nR_2^nR_4^nR_3^n\} \qquad S_3^n = \{R_1^nR_3^nR_2^nR_4^n\}$$
$$S_4^n = \{aR_1^nR_2^na^-R_3^nR_4^n\} \qquad S_5^n = \{aR_1^nR_3^na^-R_2^nR_4^n\}$$
$$S_6^n = \{aR_1^nR_4^na^-R_2^nR_3^n\} \qquad S_7^n = \{aR_1^na^-R_3^nR_2^nR_4^n\}$$
$$S_8^n = \{R_1^nR_2^naR_3^na^-bR_4^nb^-\} \qquad S_9^n = \{R_1^nR_3^naR_2^na^-bR_4^nb^-\}$$
$$S_{10}^n = \{R_1^nR_4^naR_2^na^-bR_3^nb^-\} \qquad S_{11}^n = \{R_1^naR_2^na^-bR_3^nb^-cR_4^nc^-\}$$

where $R_1^n = a_{k_1}a_{k_2}a_{k_3}\cdots a_{k_r}$, $R_2^n = a_{k_{r+1}}a_{k_{r+2}}a_{k_{r+3}}\cdots a_{k_n}$, $R_3^n = a_{t_1}^-a_{t_2}^-a_{t_3}^-\cdots a_{t_s}^-$, $R_4^n = a_{t_{s+1}}^-a_{t_{s+2}}^-a_{t_{s+3}}^-\cdots a_{t_n}^-$, $n \ge k_1 > k_2 > k_3 > \cdots > k_r \ge 1$, $1 \le k_{r+1} < k_{r+2} < k_{r+3} < \cdots < k_n \le n$, $n \ge t_1 > t_2 > t_3 > \cdots > t_s \ge 1$, $1 \le t_{s+1} < t_{s+2} < t_{s+3} < \cdots < t_n \le n$ and $1 \le r, s \le n$, $k_p \le k_q$, $t_p \ne t_q$ for $p \ne q$.

The *cross surface sets* U_k^n are as follows for $1 \le k \le 11$:

$$U_1^n = \{K_1^nK_2^nK_3^nK_4^n\} \qquad U_2^n = \{K_1^nK_2^nK_4^nK_3^n\} \qquad U_3^n = \{K_1^nK_3^nK_2^nK_4^n\}$$
$$U_4^n = \{aK_1^nK_2^na^-K_3^nK_4^n\} \qquad U_5^n = \{aK_1^nK_3^na^-K_2^nK_4^n\}$$
$$U_6^n = \{aK_1^nK_4^na^-K_2^nK_3^n\} \qquad U_7^n = \{aK_1^na^-K_3^nK_2^nK_4^n\}$$
$$U_8^n = \{K_1^nK_2^naK_3^na^-bK_4^nb^-\} \qquad U_9^n = \{K_1^nK_3^naK_2^na^-bK_4^nb^-\}$$
$$U_{10}^n = \{K_1^nK_4^naK_2^na^-bK_3^nb^-\} \qquad U_{11}^n = \{K_1^naK_2^na^-bK_3^nb^-cK_4^nc^-\}$$

where $K_1^n = a_{h_1}a_{h_2}a_{h_3}\cdots a_{h_r}$, $K_2^n = a_{h_{r+1}}a_{h_{r+2}}a_{h_{r+3}}\cdots a_{h_n}$, $K_3^n = a_{l_1}^-a_{l_2}^-a_{l_3}^-\cdots a_{l_s}^-$, $K_4^n = a_{l_{s+1}}^-a_{l_{s+2}}^-a_{l_{s+3}}^-\cdots a_{l_n}^-$, $n \ge h_1 > h_2 > h_3 > \cdots > h_r \ge 1$, $1 \le h_{r+1} < h_{r+2} < h_{r+3} < \cdots < h_n \le n$, $1 \le l_1 < l_2 < l_3 < \cdots < l_s \le n$, $n \ge l_{s+1} > l_{s+2} > l_{s+3} > \cdots > l_n \ge 1$ and $1 \le r, s \le n$, $h_p \ne h_q$, $l_p \ne l_q$ for $p \ne q$.

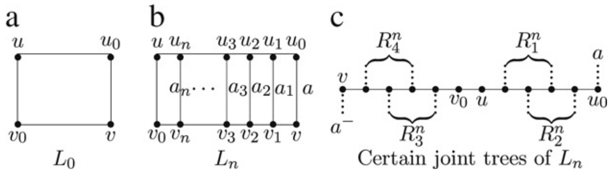

Fig. 2. L_0, L_n and certain joint trees of L_n.

Theorem 1 (*Theorem 3.1 of [7]*). *Let $g_i(GL_n)$ denote the number of distinct embeddings with genus i in GL_n and let $g_{i_j}(n)$ denote the number of surfaces with genus i in S_j^n. $g_i(G_n)$ is a linear combination of $g_{m_j}(n)$'s for $1 \le j \le 11$, $0 \le m \le i$ and $n \ge 1$.* □

Theorem 2. *Let $g_i(GC_n)$ denote the number of distinct embeddings with genus i in GC_n and let $\mu_{i_j}(n)$ denote the number of surfaces with genus i in U_j^n. $g_i(GC_n)$ is a linear combination of $\mu_{m_j}(n)$'s for $1 \le j \le 11$, $0 \le m \le i$ and $n \ge 1$.*

Proof. This conclusion holds on using arguments similar to those in the proof of Theorem 1. □

Theorem 3. *Suppose that $g_{i_j}(n)$ and $\mu_{i_j}(n)$ denote the number of surfaces for the surface sets S_j^n and U_j^n with genus i for $n \ge 1$, $1 \le j \le 11$ and $i \ge 0$ respectively. Let $f_{S_j^0}(x) = f_{U_j^0}(x) = 0$. Then,*

$$
\begin{aligned}
&g_{i_1}(n) = \mu_{i_2}(n), \quad g_{i_2}(n) = \mu_{i_1}(n), \quad g_{i_3}(n) = \mu_{i_3}(n), \quad g_{i_4}(n) = \mu_{i_4}(n),\\
&g_{i_5}(n) = \mu_{i_6}(n), \quad g_{i_6}(n) = \mu_{i_5}(n), \quad g_{i_7}(n) = \mu_{i_7}(n), \quad g_{i_8}(n) = \mu_{i_8}(n),\\
&g_{i_9}(n) = \mu_{i_{10}}(n), \quad g_{i_{10}}(n) = \mu_{i_9}(n), \quad g_{i_{11}}(n) = \mu_{i_{11}}(n).
\end{aligned}
$$

Proof. Let a_l denote distinct letters for $l \ge 1$ and let

$$
\begin{aligned}
&R_1^n = a_{k_1}a_{k_2}a_{k_3}\cdots a_{k_r}, \qquad R_2^n = a_{k_{r+1}}a_{k_{r+2}}a_{k_{r+3}}\cdots a_{k_n},\\
&R_3^n = a_{t_1}^- a_{t_2}^- a_{t_3}^- \cdots a_{t_s}^- \quad \text{and} \quad R_4^n = a_{t_{s+1}}^- a_{t_{s+2}}^- a_{t_{s+3}}^- \cdots a_{t_n}^-
\end{aligned}
$$

where $n \ge k_1 > k_2 > k_3 > \cdots > k_r \ge 1$, $1 \le k_{r+1} < k_{r+2} < k_{r+3} < \cdots < k_n \le n$, $n \ge t_1 > t_2 > t_3 > \cdots > t_s \ge 1$, $1 \le t_{s+1} < t_{s+2} < t_{s+3} < \cdots < t_n \le n$ and $1 \le r, s \le n$, $k_p \ne k_q$, $t_p \ne t_q$ for $p \ne q$. The corresponding cross surface sets U_j^n are obtained by letting

$$
\begin{aligned}
&K_1^n = a_{k_1}a_{k_2}a_{k_3}\cdots a_{k_r}, \qquad K_2^n = a_{k_{r+1}}a_{k_{r+2}}a_{k_{r+3}}\cdots a_{k_n},\\
&K_3^n = a_{t_{s+1}}^- a_{t_{s+2}}^- a_{t_{s+3}}^- \cdots a_{t_n}^- \quad \text{and} \quad K_4^n = a_{t_1}^- a_{t_2}^- a_{t_3}^- \cdots a_{t_s}^-.
\end{aligned}
$$

Let ψ be a map defined on $\bigcup_{j=1}^{11} S_j^n$ such that R_1^n, R_2^n, R_3^n and R_4^n correspond to K_1^n, K_2^n, K_4^n and K_3^n.

For any surface $R_1^nR_2^nR_3^nR_4^n \in S_1^n$, we have $\psi(R_1^nR_2^nR_3^nR_4^n) = K_1^nK_2^nK_4^nK_3^n$. For any surface $K_1^nK_2^nK_4^nK_3^n$, $\psi^{-1}(K_1^nK_2^nK_4^nK_3^n) = R_1^nR_2^nR_3^nR_4^n$. Then, ψ is a bijection from S_j^1 to U_j^2.

Since $\gamma(R_1^nR_2^nR_3^nR_4^n) = \gamma(K_1^nK_2^nK_4^nK_3^n)$, $g_{i_1}(n) = \mu_{i_2}(n)$.

The other equations can be verified by using a similar map ψ as well as by applying Lemma 1. □

3. Applications

Let L_0 be the graph shown in Fig. 2 (a). The closed-end ladder L_n is formed by adding n parallel edges u_lv_l, denoted by a_l, in Fig. 2 (b). A spanning tree T_n of L_n is obtained by letting a and a_l be cotree edges for $1 \le l \le n$. Joint trees of L_n are obtained by splitting cotree edges. Let each vertex have a clockwise rotation in each joint tree. Thus, the associated embedding surfaces of L_n are $aR_1^nR_4^na^-R_3^nR_2^n$. Certain joint trees of L_n are shown in Fig. 2 (c).
By Op2 and Lemma 1

$$aR_1^nR_4^na^-R_3^nR_2^n \sim aR_1^nR_4^na^-R_2^nR_3^n.$$

Thus,

$$f_{L_n}(x) = f_{S_6^n}(x).$$

By Theorem 3

$$f_{L_n}(x) = f_{U_5^n}(x).$$

Then, by using Theorem 4 of [10] we have:

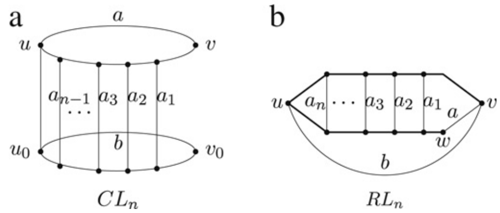

Fig. 3. CL_n and RL_n.

Corollary 1 (*[1]*). *Let $g_i(L_n)$ be the number of distinct embeddings for L_n and let $C_n(i) = \binom{n-2-i}{i}$. Then*

$$g_i(L_n) = \begin{cases} 2^{n+i-1}\dfrac{2n-3i+2}{n-i+1}C_{n+3}(i), & \text{if } 0 \le i \le \left[\dfrac{n+1}{2}\right] \text{ and } n \ge 1; \\ 0, & \text{otherwise.} \end{cases}$$

A subdivision of CL_n, still denoted by CL_n, is obtained by adding $n-1$ parallel edges a_l such that the ends of a_l are, respectively, on uv and u_0v_0 for positive integers n and l with $n \ge 2$ and $1 \le l \le n-1$ (Fig. 3 (a)). A spanning tree of CL_n is obtained by letting a, b and a_l be cotree edges. Then CL_n has four types of joint trees for certain R_1^{n-1}, R_2^{n-1}, R_3^{n-1} and R_4^{n-1} according to distinct rotation pairs at ends of a and b. Accordingly, it has four embedding surfaces $R_1^{n-1}aR_2^{n-1}a^-bR_3^{n-1}b^-R_4^{n-1}$, $a^-R_1^{n-1}aR_2^{n-1}bR_3^{n-1}b^-R_4^{n-1}$, $R_1^{n-1}aR_2^{n-1}a^-R_3^{n-1}b^-R_4^{n-1}b$ and $a^-R_1^{n-1}aR_2^{n-1}R_3^{n-1}b^-R_4^{n-1}b$. Thus,

$$g_i(CL_n) = 2g_{i_9}(n-1) + 2g_{i_{10}}(n-1) = 2\mu_{i_9}(n-1) + 2\mu_{i_{10}}(n-1).$$

By using Theorem 4 of [10], we obtain

Corollary 2 (*[5]*). *Let $g_i(CL_n)$ be the number of distinct embeddings with genus i for CL_n and let $g_0(CL_1) = 4$ where $n \ge 1$ and $i \ge 0$. Let $C_n(i) = \binom{n-2-i}{i}$ and $D_n(i) = \frac{n}{i}2^i$. Then,*

$$g_i(CL_n) = \begin{cases} 4, & \text{if } i = 0 \text{ and } n = 2; \\ 12, & \text{if } i = 1 \text{ and } n = 2; \\ 2^n + 8n + 6, & \text{if } i = 1 \text{ and } n = 3, 4; \\ 2^n + 8n - 2, & \text{if } i = 1 \text{ and } n \ge 5; \\ (2^n - 2^{2i-2})C_n(i-2)D_n(i-1) + 2^{2i}C_n(i-1)D_n(i), & \text{if } 2 \le i < \dfrac{n}{2} - 1 \text{ and } n \ge 3; \\ (2^n - 2^{2i-2})C_n(i-2)D_n(i-1) + 2^{2i}C_n(i-1)D_n(i) + 2^{n-1}, & \text{if } i = \dfrac{n}{2} - 1 \text{ and } n \ge 5; \\ (2^n - 2^{2i-2})C_n(i-2)D_n(i-1) + 2^{2i}C_n(i-1)D_n(i) + 2^n, & \text{if } \dfrac{n}{2} - 1 < i \le \dfrac{n-1}{2} \text{ and } n \ge 4; \\ (2^n - 2^{2i-2})C_n(i-2)D_n(i-1) + 2^{\frac{3n}{2}+1} - 3 \cdot 2^{n-1}, & \text{if } \dfrac{n-1}{2} < i \le \dfrac{n}{2} \text{ and } n \ge 3; \\ (2^n - 2^{2i-2})C_n(i-2)D_n(i-1), & \text{if } \dfrac{n}{2} < i \le \dfrac{n+1}{2} \text{ and } n \ge 3; \\ 0, & \text{otherwise.} \end{cases}$$

RL_n is obtained by adding parallel edges a_l such that the ends of a_l are on uv and uw each for a positive integer n and $1 \le l \le n$ (Fig. 3 (b)). Let a, b and a_l be cotree edges. A spanning tree of RL_n is obtained. RL_n has four types of embedding surfaces: $R_1^n baR_2^n b^- R_3^n a^- R_4^n$, $b^- R_1^n baR_2^n R_3^n a^- R_4^n$, $R_1^n abR_2^n b^- R_3^n a^- R_4^n$ and $b^- R_1^n abR_2^n R_3^n a^- R_4^n$.

$$g_i(RL_n) = 2g_{(i-1)_3}(n) + 2g_{i_{10}}(n) = 2\mu_{(i-1)_3}(n) + 2\mu_{i_9}(n).$$

The following conclusion can be obtained by using Theorem 4 of [10]:

Corollary 3 (*[4]*). *Let $g_i(RL_n)$ denote the number of distinct embeddings for RL_n with genus i and let $C_n(i) = \binom{n-i}{i-2}\frac{n}{i-1}2^{i-1}$. Then*

$$\begin{aligned} g_i(RL_n) = {} & 2^n C_n(i) \quad (\text{if } 1 \le i \le (n+1)/2) \\ & -2^{2i-2}C_n(i) \quad (\text{if } 2 \le i \le (n+1)/2) \\ & +2^{2i}C_n(i+1) \quad (\text{if } 1 \le i \le (n-1)/2) \\ & +2^n \quad (\text{if } i = (n-1)/2) \\ & +2^{n-1} \quad (\text{if } i = n/2 \text{ or } i = (n/2) - 1) \end{aligned}$$

$$+2^n(2^{(n/2)+1}-2) \quad (\mathit{if}\ i=n/2)$$
$$-2 \quad (\mathit{if}\ i=1)$$
$$+2 \quad (\mathit{if}\ i=0). \quad \square$$

Corollary 4 ([4]). *The embedding distribution by genus for ML_n equals that of CL_n, except that ML_n has two extra embeddings of genus* 1 *and two fewer embeddings of genus* 0. $\square$

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